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Natural extension of the Generalised Uncertainty Principle

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We discuss a gedanken experiment for the simultaneous measurement of position and momentum of a particle in de Sitter spacetime. We propose an extension of the so called Generalised Uncertainty Principle (GUP) which *implies* the existence of a minimum observable momentum. The new GUP is directly connected to the non–zero cosmological constant, which becomes a necessary ingredient for a more complete picture of the quantum spacetime.

Introduction — In quantum mechanics, two operators which do not commute cannot be measured simultaneously with arbitrary accuracy. In the case of position x and momentum along the same direction p, we obtain the well known formula

$$\Delta x \, \Delta p \gtrsim 1 \,, \tag{1}$$

where Δx and Δp denote position and momentum uncertainty respectively (we use $\hbar = c = 1$ units throughout the paper). When one takes gravitational interactions into account, eq. (1) must be revised, since the classical concept of spacetime breaks down as we approach distances close to the Planck length $L_{Pl} \sim 10^{-33}$ cm. Even if at present we do not have a fully reliable and trustworthy theory of quantum gravity, from rather general and more or less model independent considerations [1, 2, 3] one can conclude that the Uncertainty Principle (1) should be replaced by the so called Generalised Uncertainty Principle (GUP)

$$\Delta x \, \Delta p \gtrsim 1 + \alpha \, L_{Pl}^2 \, (\Delta p)^2 \,, \tag{2}$$

where α is a positive dimensionless parameter which is expected to be of order one. It is easy to see that eq. (2) implies the existence of a minimum observable length, $(\Delta x)_{min} \approx 2 \, \alpha^{1/2} \, L_{Pl}$. Among other considerations, this could be interpreted as an indication of the necessity of replacing point–like particles with extended objects in any consistent theory of quantum gravity [2], and as a signal of the breakdown of the concept of continuum spacetime at very small scales [3]. Of course, by turning off gravitational interactions eq. (2) reduces to eq. (1), since $L_{Pl} \to 0$.

However, one notes that eq. (2) is neither very "appealing" from an aesthetic point of view, nor "democratic", because the position x and the momentum p seem to play different roles: the square of Δp appears on the right hand side of eq. (2), but the square of Δx does not. Therefore, on such grounds, one is tempted to try to consider possible extensions of eq. (2). The most natural one is

$$\Delta x \, \Delta p \gtrsim 1 + \alpha \, L_{Pl}^2 \, (\Delta p)^2 \, + \beta \, \frac{(\Delta x)^2}{L_X^2} \,, \tag{3}$$

where β is a new dimensionless order one coefficient and L_X a new unknown fundamental length. This extended GUP is invariant under the following transformations

$$\begin{aligned} \Delta x \; &\rightarrow \; \left(\frac{\alpha}{\beta}\right)^{1/2} \, (L_X L_{Pl}) \Delta p \,, \\ \Delta p \; &\rightarrow \; \left(\frac{\beta}{\alpha}\right)^{1/2} \, (L_X L_{Pl})^{-1} \Delta x \,, \end{aligned} \tag{4}$$

which explicitely underline the symmetry properties of eq. (3).

Such a proposal, with $L_X = L_\Lambda$, where $L_\Lambda = (3/\Lambda)^{1/2}$ is the de Sitter horizon, had been already put forward in ref. [4], in order to obtain heuristically the temperature of (Anti–) de Sitter black holes along the same line of reasoning which permits the derivation of the Hawking temperature for Schwarzschild black holes from the standard Uncertainty Principle (1). There are, however, two important differences.

First, in ref. [4] a *positive* cosmological constant gives rise to a *negative* β, thereby preventing the interpretation of the extended GUP as an indication for the existence of a minimum observable momentum.

Second, the argument of ref. [4] has been criticised in [5], where it is shown that (Anti–) de Sitter black hole temperature can also be deduced using only eq. (1).

In this letter, by performing a simple gedanken experiment where we aim at measuring position and momentum of a particle in de Sitter spacetime, we reobtain eq. (3) with $L_X = L_\Lambda$, but now with a *positive* β: $\beta > 0$. This result differs from the previous one and, importantly, implies the existence of a theoretical bound on the minimum momentum we can measure. In particular, here we will use only quantum mechanics and general relativity, while in ref. [4] eq. (3) arises from high energy physics and the quantum structure of spacetime[1].

[1] Something similar happens for instance in the derivation of the GUP. In [1] the framework is that of string theory, whereas in ref. [3] the author relies on only quantum mechanics and general relativity arguments.

A final remark for this introduction: a relation between minimum observable momentum and cosmological constant was also conjectured from group theory arguments in the context of deformed special relativity theories in ref. [6]. Indeed, just from the definition of expectation value and unceratinty of an observable, we have

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}\,|\langle[A,B]\rangle|^2 \qquad (5)$$

where A and B are two general observables (in our case they would be x and p). So, there is always a strong connection between unceratinty relations and commutation relations. These hypothesis, however, were not followed by supportive operational derivation. Moreover, the heuristic argument we present here, although less rigorous, is more general and cautious, because it does not require a well defined framework and, as it is known, the choice of the commutation relation leading to a particular uncertainty relation is not unique. The relation between different choices are not yet completely clear [6].

The standard GUP — Let us now recall very briefly the standard experiment of the Heisenberg microscope, in which we want to measure position and momentum of a particle, say an electron. The setup of the experiment is sketched in fig. 1. Using a photon with wavelength λ, we are able to resolve at best length scales of order λ itself, so the projected electron position uncertainty is $\Delta x \gtrsim \lambda/\sin\theta$, where θ is the angle defined in fig. 1. At the same time, the interaction between the electron and the photon changes the original momentum of the former resulting in $\Delta p \gtrsim \sin\theta/\lambda$. Thus, we arrive at eq. (1).

With the same experimental setup, we can also obtain directly eq. (2), for example if we replace the electron with an extremal black hole [2]. The latter absorbs the photon with wavelength λ and then decays back to the extremal state, emitting a single photon with the same wavelength (extremal black holes are believed to be stable and thermal description breaks down for near extremal states). Before absorption and after emission, the black hole mass is M and its radius is R, whereas in the time interval between the two events the mass is $M' = M + 2\,\pi\,\lambda^{-1}$ and the radius is

$$R' \gtrsim R + 2\,G_N\,\lambda^{-1}\,. \qquad (6)$$

Since the measurement is unable to discriminate between the two radii, and $G_N = L_{Pl}^2$, we can conclude that gravity introduces an extra contribution to position uncertainty, i.e. $\Delta x_{new} \gtrsim L_{Pl}^2\,\lambda^{-1}$. The lower bound on Δx in term of $\Delta p \sim \sin\theta/\lambda$ is eq. (2).

This result has been found using a rather complicated, though explicit, setup, but it is important to notice how we could have expected it very simply on dimensional grounds. Indeed, in addition to the standard uncertainty one can reasonably argue that gravitational interaction between the photon and the particle should (weakly but inevitably) contribute. This term is independent on the nature of the probe, as follows from the Equivalence Principle. Hence, the leading order correction is expected to be proportional to G_N, which straightforwardly, for dimensional reasons, fixes the overall structure of (2).

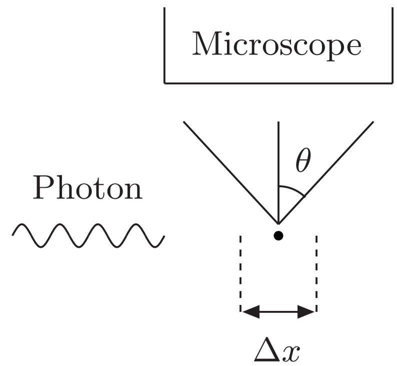

FIG. 1: Setup of the experiment. Since we do not know the exact position of the electron, when $\Lambda \neq 0$ we cannot know the exact cosmological redshift of the photon. This leads to an extra contribution to the momentum uncertainty Δp.

One even further simple and intuitive derivation is provided for example in [7]. In quantum mechanics, the accuracy on the position of a particle is limited by the uncertainty on its momentum by $\Delta x \gtrsim 1/\Delta p$. On the other hand, in general relativity we cannot localise a certain amount of energy in a region smaller than the one defined by its gravitational radius, i.e. $\Delta x \gtrsim G_N \Delta p$. If we combine the two result we conclude that

$$\Delta x \gtrsim \max(1/\Delta p\,,\, G_N \Delta p)\,.$$

The GUP then follows, upon multiplication by Δp.

The extended GUP — Now we assume to live in a de Sitter universe with cosmological constant Λ. Here the photon momentum is affected by the cosmological redshift, its magnitude depending on the path length of the photon before its detection: in particular, we should expect that an uncertainty on such a path length implies an uncertainty on the cosmological redshift and hence on p. The setup of the experiment is the same as before, the only difference is that the spacetime has a positive cosmological constant Λ.

Let us start with a very rough estimate, based on Newtonian mechanics, which reminds one of the approaches suggested in ref. [3] for the derivation of the GUP (2) and would like to show that the result comes from rather general considerations on the effects of a non–zero cosmological constant. Indeed, like in the GUP, the key-point is the validity of the Equivalence Principle, which implies that the phenomenon is independent of the experimental setup, that is of the nature of the particle and of the probe. Since the framework is non–relativistic, we replace the photon with a non–relativistic particle of mass m. It is easy to see that the Newtonian acceleration induced by a particle of mass M on a second one in a spacetime with cosmological constant Λ, is (for more details

on the derivation of this limit, see e.g. ref. [8])

$$\ddot{\mathbf{r}} = \left(-\frac{G_N M}{r^2} + \frac{\Lambda}{3}\, r\right) \frac{\mathbf{r}}{r}\,, \tag{7}$$

where $\mathbf{r}$ is the position vector of the second particle with respect to the one of mass M. The formula is obtained considering the Newtonian limit in static and spherically symmetric coordinates. Of course, eq. (7) reduces to standard Newtonian gravity for $\Lambda \to 0$. From eq. (7) we see that in our gedanken experiment there is an inevitable effective extra acceleration, due to a non–zero cosmological constant. This new contribution will affect the interaction between the electron, whose position and momentum we want to measure, and the particle we use as a probe, in particular by making the momentum of the outgoing particle uncertain. Relying once again on dimensional analysis we can infer the structure of the leading correction to be proportional to $\Lambda \sim L_\Lambda^{-2}$, and therefore to $(\Delta x)^2$.

The outlined Newtonian estimate may appear quite unreliable, or significant only in a particular limit; let us now consider a more sophisticated and complete picture. We will work in Friedman–Robertson–Walker metric with zero spatial curvature

$$ds^2 = dt^2 - a(t)^2 \left(dr^2 + r^2 d\Omega^2\right)\,, \tag{8}$$

where $a \sim \exp(Ht)$ is the cosmological scale factor. Assuming that we know exactly the electron position along the direction orthogonal to x, we send a photon towards the electron along the x direction, as shown in fig. 1. Since the position x of the point where the photon hits the electron and is then scattered, is known with a non–zero uncertainty Δx, the uncertainty in the path length of the photon before its detection by the microscope is at best at the same level, that is, Δx. This implies that the momentum of a photon is affected by the redshift logarithmic uncertainty $\Delta q/q$

$$\frac{\Delta q}{q} = \frac{\Delta \lambda}{\lambda} = \frac{\Delta a}{a} = H\,\Delta t \approx H\,\Delta x = \frac{\Delta x}{L_\Lambda}\,, \tag{9}$$

where[2] $H = (\Lambda/3)^{1/2}$ is the Hubble parameter in the de Sitter universe. Notice that the redshift acts on the momentum of the photon, not on the wavelength, whereas the astrophysical redshift is defined operatively through the latter. Here we recast the equation by means of the de Broglie wavelength in order to give a more intuitive description of the phenomenon, but the use of λ is not necessary in obtaining the final result.

Moreover, note that even though H is usually defined through the total energy density, i.e. the energy density of matter plus the one associated to Λ, here we take into account only the cosmological constant contribution, the reason being that matter energy density is spatially constant only on large scales, but most universe is basically empty and, if $\Lambda = 0$, one can always perform the experiment in a static spacetime, for example in a Schwarzschild vacuole of a Friedmann universe, where there is not expansion at all (for more details, see e.g. section 27.3 of ref. [9]). On the other hand, a true cosmological constant is spread everywhere and is present at all scales, so the experiment subjects always to the expansion of the universe. In other words, the expansion produced by a matter density can be seen as an accidental source of systematic error, depending on where the experiment is performed; the expansion caused by Λ instead is an intrinsic property of the spacetime and is inevitable.

Now, the point here is that since we do not know the exact interaction point we equivalently do not have arbitrary precision on the redshift of the photon, which then translates into a momentum uncertainty, eq. (9). We also stress here that the zeroth order relation (1) still holds, since

$$\Delta x \simeq \lambda_T \exp^{H(t_M - t_T)} / \sin\theta\,, \tag{10}$$

but also

$$\Delta q \simeq \sin\theta \exp^{-H(t_M - t_T)} / \lambda_T\,, \tag{11}$$

where λ_T is the wavelength as measured at the interaction point T (the subscripts M and T referring to quantities as measured at the microscope and at the interaction point respectively). This means that we are allowed to add to the old uncertainty principle the newly found uncertainty.

If we use the fact that $\Delta x \gtrsim \lambda$ as follows from (1), and that the errors are then to be added by quadrature, the low energy Uncertainty Principle reads

$$\Delta p \gtrsim \Delta q \gtrsim \sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{L_\Lambda^2}} \simeq \frac{1}{\Delta x} + \frac{\Delta x}{2\,L_\Lambda^2} + \ldots\,, \tag{12}$$

thereby implying the Extended Uncertainty Principle (EUP)

$$\Delta x\,\Delta p \gtrsim 1 + \beta\,\frac{(\Delta x)^2}{L_\Lambda^2}\,. \tag{13}$$

Let us gather what we have so far: by combining (13) with the high energy source of uncertainty one arrives at the formula

$$\Delta x\,\Delta p \gtrsim 1 + \alpha\,L_{Pl}^2\,(\Delta p)^2 + \beta\,\frac{(\Delta x)^2}{L_\Lambda^2}\,, \tag{14}$$

where β is a positive dimensionless coefficient which cannot be deduced from our simple considerations. The reson for this is that we have used several inequalities and trivial simplifications, although it is natural to expect

[2] We can replace Δt by Δx only in the small $H\Delta t$ limit, which is the limit of interest given today's small H.

this coefficient to be of order one - we comment on this point in the conclusions. Notice further that, as it must be, when $\Lambda \to 0$ and the de Sitter universe becomes the Minkowski spacetime, eq. (14) reduces to eq. (2).

In writing this result we have actually taken advantage of some external arguments/hints. First of all, the overall structure of the eq. (14) is inferred by dimensional (model independent) arguments, the leading order corrections being proportional to G_N and Λ. This is a very general, intuitive, and reasonable expectation.

Secondly, there is a substantial difference in the derivations of the GUP (2) and the EUP (13). Indeed, the black hole argument leading to (2) makes use of two separate and distinct gedanken experiments for the standard and high energy contributions. The way uncertainties are combined, that is, linearly, is dictated by dimensional arguments and/or group theory ones, by picking up a particular extension of the Poincaré algebra. The expression for the EUP instead results from considering one single ideal experiment, and expanding up to leading order, thereby requiring the standard statistical error treatment (quadrature). This can be traced in our statement that $\Delta x \gtrsim \lambda$, which is true only by means of the standard uncertainty relation, which we found to be valid in de Sitter spacetime.

Moreover, if we do not want to rely on the standard uncertainty priciple for the derivation of the EUP, following [7], we should write

$$\Delta p \gtrsim \max(1/\Delta x\,,\, p\Delta x/L_\Lambda)\,, \tag{15}$$

and therefore

$$\Delta x\,\Delta p \gtrsim 1 + \frac{(\Delta x)^2}{L_\Lambda}\,p \gtrsim 1 + \frac{(\Delta x)^2}{L_\Lambda}\,p_{min}\,, \tag{16}$$

where now p_{min} is the minimum observable momentum which we now try to estimate. Since $p_{min} \gtrsim \Delta p$, we can write

$$\Delta x\,p_{min} \gtrsim 1 + \frac{(\Delta x)^2}{L_\Lambda}\,p_{min} \tag{17}$$

and then

$$p_{min} \gtrsim \frac{L_\Lambda}{\Delta x\,L_\Lambda - (\Delta x)^2}\,. \tag{18}$$

Of course $p_{min} \gtrsim 0$, so $\Delta x \lesssim L_\Lambda$. Moreover, the minimum of eq. (18) is for $\Delta x \sim L_\Lambda/2$, which implies $p_{min} \gtrsim 4/L_\Lambda$, and we get the following expression for the new term of momentum uncertainty

$$\Delta p \gtrsim \frac{\Delta x}{L_\Lambda^2}\,, \tag{19}$$

which, having been obtained independently, can be linearly added to the standard Heisenberg uncertainty.

By analysing eq. (14) one is led to conclude about the minimum observable momentum. However there is a subtlety here, which is traceable in our derivation. Eq. (9) knows nothing about the possible existence of $(\Delta p)_{min}$, and it holds for $\Delta q/q < 1$, i.e. $\Delta x/L_\Lambda < 1$. The same assumption is required in the Newtonian framework as well, because in order to write eq. (7) we need $r^2/L_\Lambda^2 \ll 1$. On the other hand, in the end we assume that the new uncertainty principle has a wider range of validity and we get the prediction that there exists a minimum observable momentum for $\Delta x \sim L_\Lambda$. Let us note that a similar trick affects the standard GUP in (2), where the minimum observable position is gotten for $\Delta p \sim M_{Pl}$, but we cannot be sure that the formula still holds and does not break down earlier (for example, in some theoretical frameworks single particles cannot exceed the Planck energy). Indeed, one may expect that both the additional terms to the standard Heisenberg Uncertainty Principle (1) are only the first order correction of a more general formula which can be possibly deduced only from the true quantum theory of gravity. On the other hand, this is the drawback of all the heuristic considerations which try to investigate something beyond their realm of validity. Still, the qualitative picture should be trusted.

It is important to note that we obtain the same result even in static coordinates, where there is no true cosmological redshift. For example, if we take the de Sitter metric in static and spherically symmetric coordinates, where the line element is

$$ds^2 = \left(1 - \frac{r^2}{L_\Lambda^2}\right)dt^2 - \frac{dr^2}{1 - \frac{r^2}{L_\Lambda^2}} - r^2 d\Omega^2\,, \tag{20}$$

the photon redshift and blueshift depend only on the point of the spacetime where the photon is detected, regardless of where the collision between the photon and the electron takes place. Despite that, here we have an horizon L_Λ and the photon wavelength cannot exceed the de Sitter horizon, i.e. the effective radius of the spacetime. Hypothesising that there are no dramatic changes of the ordinary de Broglie relation up to wavelength of order L_Λ, we find $q_{min} \sim 1/L_\Lambda$, which implies and justifies the introduction of the last term of eq. (14) into the usual GUP.

Conclusions — We conclude with a few comments. First of all, from this point of view the cosmological constant Λ appears as a fundamental ingredient of the spacetime, at the same level of the gravitational constant G_N, without which we cannot regard position x and momentum p as lying on common grounds. Here we would like to emphasise once more that in our approach $\beta > 0$, yielding for the extraction of a minimum observable momentum, and restoring the symmetry between position and momentum broken by the usual GUP. Such an idea is even reinforced by present astrophysical and cosmological observations which favour a tiny but non–zero Λ responsible for the present accelerating expansion rate of the universe [10]. On the other hand, the cosmological constant remains an unexplained free parameter to be determined by experiments.

As for possible physical implications of eq. (14), one expects departures from the canonical commutation re-

lations in quantum mechanics and in quantum field theory and, as by–product, consequences in thermodynamics and statistical mechanics. The new GUP may be seen as an indication of quantum gravity deviations from the classical spacetime at both very small and very large scales (present cosmological observations suggest the value $L_\Lambda \sim 10^{28}$ cm); this may also be consistent with studies on the stability of Minkowski spacetime [11]. IR deviations from classical general relativity would represent alternative signatures in our search for evidence of the quantum structure of the spacetime and may be easier to observe than the expected UV phenomena near the Planck scale. In this connection, let us notice that the actual practical applications appear quite unlikely, which in turn implies that there are not stringent significative experimental limits on the coefficient β: the numbers we are dealing with are too small. The new picture may also (probably: only) play an important role in the physics of the early universe, during the inflationary epoch [12], where the effective cosmological constant had to be much larger than the one in the present universe, with L_Λ probably at the level of $10^{-28} - 10^{-24}$ cm. However in this case a more accurate derivation of eq. (14) is demanded, as some of our expansions and simplifications do no longer apply. We plan to study some consequences of the Extended GUP in another work.

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